Series of convex functions: subdifferential, conjugate and applications to entropy minimization

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Abstract

A formula for the subdifferential of the sum of a series of convex functions defined on a Banach space was provided by X. Y. Zheng in 1998. In this paper, besides a slight extension to locally convex spaces of Zheng's results, we provide a formula for the conjugate of a countable sum of convex functions. Then we use these results for calculating the subdifferentials and the conjugates in two situations related to entropy minimization, and we study a concrete example met in Statistical Physics.

Key words: Series of convex functions, subdifferential, conjugate, entropy minimization, statistical physics.

1 Introduction

The starting point of this study is the method used for deriving maximum entropy of ideal gases in several books dedicated to statistical physics (statistical mechanics); see [4, pp. 119, 120], [3, pp. 15, 16], [7, p. 43], [1, p. 39]. The problem is reduced to maximize $-\sum_{i\in I} n_i(\ln n_i - 1)$ [equivalently to minimize $\sum_{i\in I} n_i(\ln n_i - 1)$] with the constraints $\sum_{i\in I} n_i = N$ and $\sum_{i\in I} n_i\varepsilon_i = \varepsilon$ with n_i nonnegative integers, or, by normalization (taking $p_i := n_i/N$), to maximize $-\sum_{i\in I} p_i(\ln p_i - 1)$ [equivalently to minimize $\sum_{i\in I} p_i(\ln p_i - 1)$] with the constraints $\sum_{i\in I} p_i = 1$ and $\sum_{i\in I} p_i e_i = e$ with $p_i \in \mathbb{R}_+$. For these one uses the Lagrange multipliers method in a formal way. Even if nothing is said about the set I, from examples (see [4, (47.1)], [3, (3.11)], [7, (1.4.5)], etc) one guesses that I is a countable set. Our aim is to treat rigorously such problems. Note that the problem of minimum entropy in the case in which the infinite sum is replaced by an integral on a finite measure space and the constraints are defined by a finite number of (continuous) linear equations is treated rigorously by J. M. Borwein and his collaborators in several papers (in the last 25 years); see [2] for a recent survey.

The plan of the paper is the following. In Section 2 we present a slight extension (to locally convex spaces) of the results of X. Y. Zheng [11] related to the subdifferential of the sum of a series of convex functions; we provide the proofs for readers convenience. In Section 3 we apply the results in Section 2 for deriving a formula for the conjugate of the sum of a series of convex functions, extending so Moreau's Theorem on the conjugate of the sum of a finite family of convex functions to countable sums of such functions. In Section 4 we apply the results in the preceding sections to find the minimum entropy for a concrete situation from Statistical Physics.

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2 Series of convex functions

Throughout this paper, having a sequence $(A_n)_{n\geq 1}$ of nonempty sets and a sequence $(x_n)_{n\geq 1}$, the notation $(x_n)_{n\geq 1} \subset (A_n)_{n\geq 1}$ means that $x_n \in A_n$ for every $n \geq 1$.

In the sequel (E, τ) is a real separated locally convex space (lcs for short) and E^* is its topological dual. Moreover, we shall use standard notations and results from convex analysis (see e.g. [6], [10]). Consider $f_n \in \Lambda(E)$ (that is f_n is proper and convex) for every $n \geq 1$. Assume that $f(x) := \sum_{n\geq 1} f_n(x) := \lim_{n\to\infty} \sum_{k=1}^n f_k(x)$ exists in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ for every $x \in E$, where $\infty := +\infty$. Then, clearly, the corresponding function $f : E \to \overline{\mathbb{R}}$ is convex and $\operatorname{dom} f \subset \cap_{n\geq 1} \operatorname{dom} f_n$.

Note that, if $(f_n)_{n\geq 1} \subset \Gamma(E)$ (that is $f_n \in \Lambda(X)$ is also lower semicontinuous, lsc for short) and there exists $(x_n^*)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1}$ such that the series $\sum_{n\geq 1} f_n^*(x_n^*)$ is convergent and $w^*\text{-}\lim_{n\to\infty} \sum_{k=1}^n x_k^* = x^* \in E^*$ (that is $x^* = w^*\text{-}\sum_{n\geq 1} x_n^*$), then $\lim_{n\to\infty} \sum_{k=1}^n f_k(x)$ exists and belongs to $(-\infty, \infty]$ for every $x \in E$; moreover, f is lsc.

Indeed, since $g_n(x) := f_n(x) + f_n^*(x_n^*) - \langle x, x_n^* \rangle \geq 0$, $g(x) := \lim_{n\to\infty} \sum_{k=1}^n g_n(x) = \sup_{n\geq 1} g_n(x)$ exists and belongs to $[0, \infty]$. But $\sum_{k=1}^n g_n(x) = \sum_{k=1}^n f_n(x) + \sum_{k=1}^n f_n^*(x_n^*) - \langle x, \sum_{k=1}^n x_n^* \rangle$, and $\gamma := \lim_{n\to\infty} \sum_{k=1}^n f_n^*(x_n^*) \in \mathbb{R}$, $\lim_{n\to\infty} \langle x, \sum_{k=1}^n x_n^* \rangle = \langle x, x^* \rangle$. It follows that $f(x) = \lim_{n\to\infty} \sum_{k=1}^n f_n(x) = g(x) - \gamma + \langle x, x^* \rangle \in (-\infty, \infty]$. Since $g_n \in \Gamma(X)$ for every n and $g = \sup_{n\geq 1} g_n$ is lsc, it follows that f is lsc, too.

Definition 1 (Zheng [11, p. 79]) *Let $A, A_n \in \mathcal{P}_0(E) := \{F \subset E \mid F \neq \emptyset\}$ $(n \geq 1)$. One says that $(A_n)_{n\geq 1}$* converges normally *to A (with respect to τ), written $A = \tau\text{-}\sum_{n\geq 1} A_n$, if:*

(I) for every sequence $(x_n)_{n\geq 1} \subset (A_n)_{n\geq 1}$, the series $\sum_{n\geq 1} x_n$ τ-converges and its sum x belongs to A;

(II) for each (τ-)neighborhood U of 0 in E (that is $U \in \mathcal{N}_E^\tau$) there is $n_0 \geq 1$ such that $\sum_{k\geq n} x_k \in U$ for all sequences $(x_n)_{n\geq 1} \subset (A_n)_{n\geq 1}$ and all $n \geq n_0$ (observe that the series $\sum_{k\geq n} x_k$ is τ-convergent by (I));

(III) for each $x \in A$ there exists $(x_n)_{n\geq 1} \subset (A_n)_{n\geq 1}$ such that $x = \tau\text{-}\sum_{n\geq 1} x_n$.

Observe that A in the above definition is unique; moreover, A is convex if all A_n are convex.

Remark 2 *1) Assume that E is the topological dual X^* of the lcs X endowed with the weak* topology w^*, and $(A_n)_{n\geq 1} \subset \mathcal{P}_0(X^*)$ is such that (I) holds. Then (II) in Definition 1 holds if and only if for every $\varepsilon > 0$ and every $x \in X$ there exists $n_0 = n_{\varepsilon,x} \geq 1$ such that $\left|\sum_{k\geq n} \langle x, x_k^* \rangle\right| \leq \varepsilon$ for all sequences $(x_n^*)_{n\geq 1} \subset (A_n)_{n\geq 1}$ and all $n \geq n_0$.*

2) Assume that E is a normed vector space (nvs for short) endowed with the strong (norm) topology s, and $(A_n)_{n\geq 1} \subset \mathcal{P}_0(E)$ is such that (I) holds. Then (II) in Definition 1 holds if and only if for every $\varepsilon > 0$ there exists $n_\varepsilon \geq 1$ such that $\left\|\sum_{k\geq n} x_k\right\| \leq \varepsilon$ for all sequences $(x_n)_{n\geq 1} \subset (A_n)_{n\geq 1}$ and all $n \geq n_\varepsilon$. It follows that $A = s\text{-}\sum_{n\geq 1} A_n$ implies that A is a Hausdorff-Pompeiu limit of $(\sum_{k=1}^n A_k)_{n\geq 1}$.

In the rest of this section we mainly reformulate the results of Zheng [11] in the context of locally convex spaces without asking the functions be lower semicontinuous. We give the proofs for reader's convenience.

Theorem 3 *Let $f, f_n \in \Lambda(E)$ be such that $f(x) = \sum_{n\geq 1} f_n(x)$ for every $x \in E$. Assume that $\overline{x} \in \operatorname{core}(\operatorname{dom} f)$. Then*

$$f'_+(\overline{x}, u) = \sum_{n\geq 1} f'_{n+}(\overline{x}, u) \quad \forall u \in E.$$

Proof. Consider $u \in E$. Because $\overline{x} \in \operatorname{core}(\operatorname{dom} f)$, there exists $\delta > 0$ such that $\overline{x} + tu \in \operatorname{dom} f \subset \operatorname{dom} f_n$ for every $t \in I := [-\delta, \delta]$. Consider $\varphi, \varphi_n : I \to \mathbb{R}$ defined by $\varphi(t) := f(\overline{x} + tu)$, $\varphi_n(t) := f_n(\overline{x} + tu)$; φ, φ_n are convex and $\varphi(t) = \sum_{n\geq 1} \varphi_n(t)$ for every $t \in I$. Of course, $f'_+(\overline{x}, u) = \lim_{t\to 0+} \frac{\varphi(t)-\varphi(0)}{t}$, and similarly for $f'_{n+}(\overline{x}, u)$. Since the mappings $I \setminus \{0\} \ni t \mapsto t^{-1}\varphi_n(t) \in \mathbb{R}$ are nondecreasing we get

$$\frac{\varphi_n(-\delta) - \varphi_n(0)}{-\delta} \leq \frac{\varphi_n(t) - \varphi_n(0)}{t} \leq \frac{\varphi_n(\delta) - \varphi_n(0)}{\delta},$$

whence

$$0 \leq \psi_n(t) := \frac{\varphi_n(t) - \varphi_n(0)}{t} - \frac{\varphi_n(-\delta) - \varphi_n(0)}{-\delta} \leq \frac{\varphi_n(\delta) - \varphi_n(0)}{\delta} - \frac{\varphi_n(-\delta) - \varphi_n(0)}{-\delta} =: \gamma_n$$

for all $n \geq 1$ and $t \in (0, \delta]$. Since the series $\sum_{n\geq 1} \gamma_n$ is convergent, the series $\sum_{n\geq 1} \psi_n$ is uniformly convergent on $(0, \delta]$. It follows that

$$\lim_{t\to 0+} \sum_{n\geq 1} \psi_n(t) = \sum_{n\geq 1} \lim_{t\to 0+} \psi_n(t).$$

Since $\sum_{n\geq 1} \frac{\varphi_n(-\delta)-\varphi_n(0)}{-\delta} = \frac{\varphi(-\delta)-\varphi(0)}{-\delta}$, we obtain that

$$f'_+(\overline{x}, u) = \lim_{t\to 0+} \frac{\varphi(t) - \varphi(0)}{t} = \lim_{t\to 0+} \sum_{n\geq 1} \frac{\varphi_n(t) - \varphi_n(0)}{t} = \sum_{n\geq 1} \lim_{t\to 0+} \frac{\varphi_n(t) - \varphi_n(0)}{t}$$
$$= \sum_{n\geq 1} f'_{n+}(\overline{x}, u).$$

The proof is complete. □

Proposition 4 *Let $f, f_n \in \Lambda(E)$ be such that $f(x) = \sum_{n\geq 1} f_n(x)$ for every $x \in E$. Assume that the series $\sum_{n\geq 1} f_n$ converges uniformly on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} f)$. Then for every $x \in \operatorname{int}(\operatorname{dom} f)$ there exists a neighborhood of x on which the series $\sum_{n\geq 1} f_n$ converges uniformly.*

Proof. Replacing (if necessary) f_n by g_n defined by $g_n(x) := f_n(x_0 + x) - f_n(x_0)$ and f by g defined by $g(x) := f(x_0 + x) - f(x_0)$, we may (and do) assume that $x_0 = 0$ and $f_n(0) = f(0) = 0$. There exists a closed, convex and symmetric neighborhood V of $x_0 = 0$ such that $2V \subset \operatorname{dom} f$ and the series $\sum_{n\geq 1} f_n$ converges uniformly on $2V$. Set $p := p_V$, the Minkowski functional associated to V. Then p is a continuous seminorm such that $\operatorname{int} V = \{x \in E \mid p(x) < 1\}$ and $\operatorname{cl} V = V = \{x \in E \mid p(x) \leq 1\}$. Consider $x \in \operatorname{int}(\operatorname{dom} f)$. If $p(x) < 2$ then $x \in \operatorname{int}(2V)$; take $U := 2V$ in this case.

Let $p(x) \geq 2$. Since $x \in \operatorname{int}(\operatorname{dom} f)$, there exists $\mu > 0$ such that $x' := (1 + \mu)x \in \operatorname{dom} f$, and so $x = (1 - \lambda)x' + \lambda 0$, where $\lambda := \mu/(1 + \mu) \in (0, 1)$. Fix $u \in V$ ($\Leftrightarrow p(u) \leq 1$); we have that $x + \lambda u = (1 - \lambda)x' + \lambda u$, and so

$$f_n(x + \lambda u) \leq (1 - \lambda)f_n(x') + \lambda f_n(u) \quad \forall n \geq 1. \tag{1}$$

On the other hand $1 < 2-\lambda \le p(x) - p(\lambda u) \le p(x+\lambda u)$, and so $\frac{x+\lambda u}{p(x+\lambda u)} \in V$ and

$$f_n\left(\frac{x+\lambda u}{p(x+\lambda u)}\right) \le \frac{1}{p(x+\lambda u)} f_n(x+\lambda u),$$

whence

$$f_n(x+\lambda u) \ge p(x+\lambda u) f_n\left(\frac{x+\lambda u}{p(x+\lambda u)}\right) \quad \forall n \ge 1. \tag{2}$$

From (1) and (2) we get

$$p(x+\lambda u)\sum_{k=l}^{m} f_n\left(\frac{x+\lambda u}{p(x+\lambda u)}\right) \le \sum_{k=l}^{m} f_k(x+\lambda u) \le (1-\lambda)\sum_{k=l}^{m} f_k(x') + \lambda \sum_{k=l}^{m} f_k(u),$$

whence, because $p(x+\lambda u) \le p(x)+1$,

$$\left|\sum_{k=l}^{m} f_k(x+\lambda u)\right| \le \left|\sum_{k=l}^{m} f_k(x')\right| + \left|\sum_{k=l}^{m} f_k(u)\right| + (p(x)+1)\left|\sum_{k=l}^{m} f_n\left(\frac{x+\lambda u}{p(x+\lambda u)}\right)\right|$$

for all $l, m \ge 1$ with $l \le m$. Since $u, \frac{x+\lambda u}{p(x+\lambda u)} \in V \subset 2V$, from the uniform convergence of $\sum_{n\ge1} f_n$ on $2V$ and the convergence of $\sum_{n\ge1} f_n(x')$, the previous inequality shows that $\sum_{n\ge1} f_n$ is uniformly convergent on $U := x + \lambda V$ ($\subset \operatorname{dom} f$). The proof is complete. $\square$

Theorem 5 *Let $f, f_n \in \Lambda(E)$ be such that $f(x) = \sum_{n\ge1} f_n(x)$ for every $x \in E$. Assume that the series $\sum_{n\ge1} f_n$ converges uniformly on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} f)$. Then for every $x \in \operatorname{int}(\operatorname{dom} f)$ there exists a neighborhood U of $0 \in E$ with $x + U \subset \operatorname{dom} f$ such that the series $\sum_{n\ge1} f'_{n+}(\cdot,\cdot)$ converges uniformly [to $f'_+(\cdot,\cdot)$] on $(x+U)\times U$.*

Proof. Taking into account Proposition 4, it is sufficient to prove the conclusion for $x = x_0$. Moreover, as in the proof of Proposition 4, we may (and do) assume that $x_0 = 0$ and $f_n(0) = 0 = f(0)$ for every $n \ge 1$. By hypothesis, there exists a convex neighborhood V of $x_0 = 0$ such that $2V \subset \operatorname{dom} f$ and the series $\sum_{n\ge1} f_n$ converges uniformly on $2V$. For $(x,h) \in V \times V$ we have that $x \pm h \in 2V$. It follows that the series $\sum_{n\ge1}[f_n(x) - f_n(x-u)]$ and $\sum_{n\ge1}[f_n(x+u) - f_n(x)]$ converge to $f(x) - f(x-u)$ and $f(x+u) - f(x)$, uniformly for $(x,u) \in V \times V$, respectively. We have that

$$f_n(x) - f_n(x-u) \le \frac{f_n(x+tu) - f_n(x)}{t} \le f_n(x+u) - f_n(x) \quad \forall t \in (0,1],\ \forall n \ge 1.$$

Hence

$$\left|\sum_{k=l}^{m} \frac{f_n(x+tu) - f_n(x)}{t}\right| \le \left|\sum_{k=l}^{m} [f_n(x+u) - f_n(x)]\right| + \left|\sum_{k=l}^{m} [f_n(x) - f_n(x-u)]\right|$$

for all $(x,u) \in V \times V$, all $t \in (0,1]$ and all $l, m \ge 1$ with $l \le m$. Using (in both senses) the Cauchy criterion, we obtain that the series $\sum_{n\ge1} \frac{f_n(x+tu)-f_n(x)}{t}$ converges uniformly for $(x,u,t) \in V \times V \times (0,1]$ to $\frac{f(x+tu)-f(x)}{t}$. Letting $t \to 0+$, we obtain that the series $\sum_{n\ge1} f'_{n+}(x,u)$ converges uniformly for $(x,u) \in V \times V$ to $f'_+(x,u)$. The proof is complete. $\square$

Remark 6 *Note that for f, f_n as in the preceding theorem, the series $\sum_{n\geq 1} f'_{n+}(\cdot,\cdot)$ converges uniformly [to $f'_+(\cdot,\cdot)$] on $(x+U)\times U$ if and only if for every $\alpha > 0$, the series $\sum_{n\geq 1} f'_{n+}(\cdot,\cdot)$ converges uniformly [to $f'_+(\cdot,\cdot)$] on $(x+U)\times(\alpha U)$.*

Lemma 7 *Let $I \subset \mathbb{R}$ be an open interval and $\varphi, \varphi_n : I \to \mathbb{R}$ $(n \geq 1)$ be nondecreasing functions such that $\varphi(t) = \sum_{n\geq 1} \varphi_n(t)$ for every $t \in I$. Then $\sum_{n\geq 1} \int_\alpha^\beta \varphi_n(t)dt = \int_\alpha^\beta \varphi(t)dt$ for all $\alpha, \beta \in I$ with $\alpha < \beta$.*

Proof. Fix $\alpha, \beta \in I$ with $\alpha < \beta$. Take $\psi, \psi_n : I \to \mathbb{R}$ $(n \geq 1)$ defined by $\psi(t) := \varphi(t) - \varphi(\alpha)$ and $\psi_n(t) := \varphi_n(t) - \varphi_n(\alpha)$ for $t \in J$. Then, clearly, ψ, ψ_n are nondecreasing functions, $\psi_n(t) \geq \psi_n(\alpha) = 0$ for $t \in J_0 := [\alpha, \beta]$ and $n \geq 1$ and

$$\sum_{n\geq 1} \psi_n(t) = \sum_{n\geq 1} [\varphi_n(t) - \varphi_n(\alpha)] = \sum_{n\geq 1} \varphi_n(t) - \sum_{n\geq 1} \varphi_n(\alpha) = \varphi(t) - \varphi(\alpha) = \psi(t)$$

for all $t \in I \supset J_0$. Since $0 \leq \psi_k$ on J_0, $\lim_{n\to\infty} \sum_{k=1}^n \psi_k(t) = \psi(t)$ for every $t \in J_0$, and ψ is Lebesgue integrable on J_0, we have that

$$\int_{J_0} \psi(t)dt = \int_{J_0} \left(\sum_{n\geq 1} \psi_n(t)\right) dt = \sum_{n\geq 1} \int_{J_0} \psi_n(t)dt.$$

But $\int_{J_0} \psi_n(t)dt = \int_{J_0} (\varphi_n(t) - \varphi_n(\alpha))\, dt = \int_\alpha^\beta \varphi_n(t)dt - (\beta - a)\varphi_n(\alpha)$, and similarly for ψ and φ, whence

$$\begin{aligned}\int_\alpha^\beta \varphi(t)dt - (\beta - a)\varphi(\alpha) &= \sum_{n\geq 1} \left(\int_\alpha^\beta \varphi_n(t)dt - (\beta - a)\varphi_n(\alpha)\right) \\ &= \sum_{n\geq 1} \int_\alpha^\beta \varphi_n(t)dt - (\beta - a) \sum_{n\geq 1} \varphi_n(\alpha),\end{aligned}$$

and so $\int_\alpha^\beta \varphi(t)dt = \sum_{n\geq 1} \int_\alpha^\beta \varphi_n(t)dt$. □

Proposition 8 *Let $f, f_n \in \Lambda(E)$. Suppose that* $\operatorname{dom} f \subset \cap_{n\geq 1} \operatorname{dom} f_n$, *and* $x_0 \in \operatorname{int}(\operatorname{dom} f)$ *is such that $f(x_0) = \sum_{n\geq 1} f_n(x_0)$.*

(i) *Assume that $\sum_{n\geq 1} f'_{n+}(x, u) = f'_+(x, u)$ for all $x \in \operatorname{int}(\operatorname{dom} f)$ and $u \in E$. Then $f(x) = \sum_{n\geq 1} f_n(x)$ for every $x \in \operatorname{int}(\operatorname{dom} f)$.*

(ii) *Moreover, assume that there exists a neighborhood U of $0 \in E$ such that $x_0 + U \subset$ $\operatorname{int}(\operatorname{dom} f)$ and the series $\sum_{n\geq 1} f'_{n+}(\cdot,\cdot)$ converges uniformly on $(x_0+U)\times U$. Then for every $x \in \operatorname{int}(\operatorname{dom} f)$ the series $\sum_{n\geq 1} f_n$ converges uniformly to f on some neighborhood of x.*

Proof. (i) Replacing, if necessary, f by g defined by $g(x) := f(x_0 + x) - f(x_0)$, and similarly for f_n, we may (an do) assume that $x_0 = 0$ and $f_n(x_0) = f(0) = 0$ for every $n \geq 1$.

Fix $x \in \operatorname{int}(\operatorname{dom} f)$. Consider $I := \{t \in \mathbb{R} \mid tx \in \operatorname{int}(\operatorname{dom} f)\}$. Then I is an open interval and $0, 1 \in I$. Take $\theta(t) := f(tx)$ and $\theta_n(t) := f_n(tx)$ for $t \in I$. Then θ, θ_n are finite and convex on I and $\theta(0) = \theta_n(0) = 0$. Moreover, $\theta'_+(t) = f'_+(tx, x)$ for every $t \in I$, and similarly for $\theta'_{n+}(t)$. Of course, θ and θ_n are nondecreasing and finite on I. Using our hypothesis, we

have that $\sum_{n\geq 1} \theta'_{n+}(t) = \theta'_+(t)$ for every $t \in I$. Using Lemma 7 with $\varphi_n = \theta'_{n+}$ and $\varphi = \theta'_+$ we obtain that

$$f(x) = \theta(1) = \int_0^1 \theta'_+(t)dt = \sum_{n\geq 1} \int_0^1 \theta'_{n+}(t)dt = \sum_{n\geq 1} \theta_n(1) = \sum_{n\geq 1} f_n(x).$$

(ii) From (i) we have that $f(x) = \sum_{n\geq 1} f_n(x)$ for every $x \in \operatorname{int}(\operatorname{dom} f)$. Taking into account Proposition 4, it is sufficient to show the conclusion for x_0. As above, we may (and do) assume that $x_0 = 0$ and $f_n(0) = f(0) = 0$. By our hypothesis, for every $\varepsilon > 0$ there exists $n_\varepsilon \geq 0$ such that

$$f'_+(x,u) - \varepsilon \leq \sum_{k=1}^{n} f'_{k+}(x,u) \leq f'_+(x,u) + \varepsilon \quad \forall (x,u) \in U \times U,\ \forall n \geq n_\varepsilon.$$

In particular,

$$f'_+(tx,x) - \varepsilon \leq \sum_{k=1}^{n} f'_{k+}(tx,x) \leq f'_+(tx,x) + \varepsilon \quad \forall t \in [0,1],\ \forall x \in U,\ \forall n \geq n_\varepsilon.$$

Integrating on $[0,1]$ with respect to t, we get $f(x) - \varepsilon \leq \sum_{k=1}^{n} f_k(x) \leq f(x) - \varepsilon$ for all $x \in U$ and $n \geq n_\varepsilon$. This shows that the conclusion holds for x_0. The proof is complete. $\square$

Theorem 9 *Let $f, f_n \in \Lambda(E)$. Assume that $f(x) = \sum_{n\geq 1} f_n(x)$ for every $x \in E$. If f and f_n are continuous on* $\operatorname{int}(\operatorname{dom} f)$*, then*

$$\partial f(x) = w^*\text{-}\sum_{n\geq 1} \partial f_n(x) \quad \forall x \in \operatorname{int}(\operatorname{dom} f).$$

Proof. (I) Fix $x \in \operatorname{int}(\operatorname{dom} f)$. Consider $(x_n^*)_{n\geq 1} \subset (\partial f_n(x))_{n\geq 1}$. We claim that the series $\sum_{n\geq 1} x_n^*$ is w^*-convergent to some $x^* \in \partial f(x)$.

Fix some $u \in E$. By Theorem 3, for every $\varepsilon > 0$, there exists $n_\varepsilon \geq 1$ such that $\left|\sum_{k=l}^{m} f'_{k+}(x, \pm u)\right| \leq \varepsilon/2$ for all $l, m \geq n_\varepsilon$ with $l \leq m$. Since $x_k^* \in \partial f_k(x)$, we have that

$$\langle \pm u, x_k^* \rangle \leq f'_{k+}(x, \pm u), \tag{3}$$

and so

$$\pm \sum_{k=l}^{m} \langle u, x_k^* \rangle \leq \sum_{k=l}^{m} f'_{k+}(x, \pm u) \leq \left| \sum_{k=l}^{m} f'_{k+}(x,u) \right| + \left| \sum_{k=l}^{m} f'_{k+}(x,-u) \right|.$$

Hence

$$\left| \sum_{k=l}^{m} \langle u, x_k^* \rangle \right| \leq \left| \sum_{k=l}^{m} f'_{k+}(x,u) \right| + \left| \sum_{k=l}^{m} f'_{k+}(x,-u) \right| \leq \varepsilon \quad \forall l, m \geq 1,\ n_\varepsilon \leq l \leq m. \tag{4}$$

Therefore, the series $\sum_{n\geq 1} \langle u, x_n^* \rangle$ is convergent, and so $\varphi(u) := \sum_{n\geq 1} \langle u, x_k^* \rangle \in \mathbb{R}$. Moreover, from (3) and Theorem 3 we get

$$\varphi(u) \leq \sum_{n\geq 1} f'_{n+}(x,u) = f'_+(x,u).$$

We got so a linear mapping $\varphi : E \to \mathbb{R}$ such that $\varphi \leq f'_+(x,\cdot)$. Since f is continuous at $x \in \operatorname{dom} f$, $f'_+(x,\cdot)$ is continuous, and so $\varphi \in \partial f(x)$. Hence w^*-$\sum_{n\geq 1} x_n^*$ exists and belongs to $\partial f(x)$. Therefore, condition (I) in Definition 1 holds.

(II) Taking the limit for $m \to \infty$ in (4) we obtain that $\left|\sum_{k\geq n} \langle u, x_k^*\rangle\right| \leq \varepsilon$ for all $n \geq n_\varepsilon$. Since n_ε does not depend on the sequence $(x_n^*)_{n\geq 1} \subset (\partial f_n(x))_{n\geq 1}$, the second condition in Definition 1 holds, too.

(III) For $n \geq 1$ set $R_n := \sum_{k\geq n} f_k$; clearly $R_n \in \Lambda(E)$ for $n \geq 1$. Using Theorem 3 for the sequence $(f_{k+n})_{k\geq 0}$ we obtain that $R'_{n+}(x,u) = \sum_{k\geq n} f'_{k+}(x,u) \to 0$ (as $n \to \infty$) for every $u \in E$. Moreover, $R_1 = f$ and $R_k = f_k + R_{k+1}$ for $k \geq 1$. Since f_k is continuous on $\operatorname{int}(\operatorname{dom} f_k) \supset \operatorname{int}(\operatorname{dom} f) = \operatorname{int}(\operatorname{dom} R_k)$, by Moreau-Rockafellar Theorem one has

$$\partial R_k(x) = \partial f_k(x) + \partial R_{k+1}(x) \quad \forall k \geq 1. \tag{5}$$

Consider $x^* \in \partial f(x) = \partial R_1(x)$. From (5) applied for $k = 1$ we get $x_1^* \in \partial f_1(x)$ such that $x^* - x_1^* \in \partial R_2(x)$. Continuing in this way we get a sequence $(x_n^*)_{n\geq 1} \subset (\partial f_n(x))_{n\geq 1}$ such that

$$y_n^* := x^* - \sum_{k=1}^{n-1} x_k^* \in \partial R_n(x) \quad \forall n \geq 1,$$

where $\sum_{k=1}^{0} x_k^* := 0$. Using Theorem 3 for the sequence $(f_{k+n})_{k\geq 0}$ and the fact that $\langle \pm u, y_n^*\rangle \leq R'_{n+}(x, \pm u)$ we obtain that

$$\left|\left\langle u, x^* - \sum_{k=1}^{n} x_k^* \right\rangle\right| = |\langle u, y_n^*\rangle| \leq \left|R'_{n+}(x,u)\right| + \left|R'_{n+}(x,-u)\right| \to 0 \text{ for } n \to \infty$$

for every $u \in E$. It follows that $x^* = w^*$-$\sum_{n\geq 1} x_n^*$. Hence condition (III) of Definition 1 is verified, too. The proof is complete. $\square$

When E is a normed vector space, one has also the next result.

Theorem 10 *Let E be a normed vector space and $f, f_n \in \Lambda(E)$. Assume that $f(x) = \sum_{n\geq 1} f_n(x)$ for every $x \in E$. If f and f_n are continuous on $\operatorname{int}(\operatorname{dom} f)$ and the series $\sum_{n\geq 1} f_n$ converges uniformly on a nonempty open subset of $\operatorname{dom} f$, then*

$$\partial f(x) = \|\cdot\| \text{-} \sum_{n\geq 1} \partial f_n(x) \quad \forall x \in \operatorname{int}(\operatorname{dom} f);$$

moreover, $\lim_{n\to\infty} \left\| \sum_{k\geq n} \partial f_k(x)\right\| = 0$ uniformly on some neighborhood of x for every $x \in \operatorname{int}(\operatorname{dom} f)$, where $\|A\| := \sup\{\|x^*\| \mid x^* \in A\}$ for $\emptyset \neq A \subset X^*$.

Proof. We follow the same steps as in the proof of Theorem 9. So, fix $x_0 \in \operatorname{int}(\operatorname{dom} f)$.

(I) Consider $(x_n^*)_{n\geq 1} \subset (\partial f_n(x_0))_{n\geq 1}$. By Theorem 9 we have that w^*-$\sum_{n\geq 1} x_n^* = x^* \in \partial f(x_0)$. We claim that $x^* = \|\cdot\|$-$\sum_{n\geq 1} x_n^*$. Indeed, using this time Proposition 4, Theorem 5 and Remark 6, there exists $r > 0$ such that the series $\sum_{n\geq 1} f'_{n+}(\cdot,\cdot)$ converges uniformly to $f'_+(\cdot,\cdot)$ on $(x_0 + rU_E) \times U_E$, where U_E is the closed unit ball of E. Taking $\varepsilon > 0$, as in the proof of Theorem 9, there exists $n_\varepsilon \geq 1$ such that (4) holds for all $(x,u) \in (x_0 + rU_E) \times U_E$, and so $\|\sum_{k=l}^{m} x_k^*\| \leq \varepsilon$ for all $n_\varepsilon \leq l \leq m$. Since $x^* = w^*$-$\sum_{n\geq 1} x_n^*$ we get $x^* = \|\cdot\|$-$\sum_{n\geq 1} x_n^*$. In fact we even get $\lim_{n\to\infty} \left\| \sum_{k\geq n} \partial f_k(x)\right\| = 0$ uniformly on $x_0 + rU_E$.

(II) This step is practically stated in (I).

(III) Having $x^* \in \partial f(x_0)$, from Theorem 9 we find $(x_n^*)_{n\geq 1} \subset (\partial f_n(x_0))_{n\geq 1}$ such that $x^* = w^*$-$\sum_{n\geq 1} x_n^*$. From (I) we obtain that $x^* = \|\cdot\|$-$\sum_{n\geq 1} x_n^*$. The proof is complete. $\square$

Corollary 11 *Let $f, f_n \in \Lambda(E)$. Assume that $f(x) = \sum_{n\geq 1} f_n(x)$ for every $x \in E$, and f, f_n are continuous on* $\operatorname{int}(\operatorname{dom} f)$ *for every $n \geq 1$. Take $\overline{x} \in \operatorname{int}(\operatorname{dom} f)$. Then*

(i) *f is Gâteaux differentiable at $\overline{x}$ if and only if f_n is Gâteaux differentiable at $\overline{x}$ for every $n \geq 1$, in which case $\nabla f(\overline{x}) = \sum_{n\geq 1} \nabla f_n(\overline{x})$.*

(ii) *Moreover, assume that E is a normed vector space. If f is Fréchet differentiable at $\overline{x}$ then f_n is Fréchet differentiable at $\overline{x}$ for every $n \geq 1$.*

Proof. By Theorem 9 we have that $\partial f(\overline{x}) = w^*\text{-}\sum_{n\geq 1} \partial f_n(\overline{x})$. This relation shows that $\partial f(\overline{x})$ is a singleton if and only if $\partial f_n(\overline{x})$ is a singleton for every $n \geq 1$. Since the functions f and f_n $(n \geq 1)$ are continuous at $\overline{x}$, (i) follows.

(ii) Assume now that E is a nvs and f is Fréchet differentiable at $\overline{x}$. It is known that g and h are Fréchet differentiable at $\overline{x} \in \operatorname{int}(\operatorname{dom} g \cap \operatorname{dom} h) = \operatorname{int}(\operatorname{dom}(g+h))$ provided $g, h \in \Lambda(E)$, g, h are continuous at $\overline{x}$ and $g + h$ is Fréchet differentiable at $\overline{x}$. This is due to the fact that in such conditions, as seen from (i), g and h are Gâteaux differentiable at $\overline{x}$. Then we have

$$0 \leq \frac{g(\overline{x}+u) - g(\overline{x}) - \langle u, \nabla g(\overline{x})\rangle}{\|u\|} \leq \frac{(g+h)(\overline{x}+u) - (g+h)(\overline{x}) - \langle u, \nabla(g+h)(\overline{x})\rangle}{\|u\|} \to 0$$

for $\|u\| \to 0$.

With the notation in the proof of Theorem 9, $f = R_1 = f_1 + R_2$. It follows that f_1 and R_2 are Fréchet differentiable at $\overline{x}$. Since $R_2 = f_2 + R_3$, it follows that f_2 and R_3 are Fréchet differentiable at $\overline{x}$. Continuing in this way we obtain that f_n is Fréchet differentiable at $\overline{x}$ for every $n \geq 1$. □

Of course, if $\dim E < \infty$, the weak* and strong convergences on E^* coincide, and so Theorems 9 and 10 are equivalent; moreover, Gâteaux and Fréchet differentiability for convex functions coincide, and so the converse implication in Corollary 11 (ii) is true.

Question 1 *Is the converse of Corollary 11 (ii) true when $\dim E = \infty$?*

In the sequel we set $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_+^* := (0, \infty)$, $\mathbb{R}_- := -\mathbb{R}_+$, $\mathbb{R}_-^* := -\mathbb{R}_+^*$.

Proposition 12 *Let $f_n(x) := e^{\sigma_n x}$ for $x \in \mathbb{R}$ with $(\sigma_n)_{n\geq 1} \subset \mathbb{R}$; set $f = \sum_{n\geq 1} f_n$.*

(i) *If $\overline{x} \in \operatorname{dom} f$ then $\sigma_n \overline{x} \to -\infty$, and so either $\overline{x} > 0$ and $\sigma_n \to -\infty$, or $\overline{x} < 0$ and $\sigma_n \to \infty$.*

(ii) *Assume that $0 < \sigma_n \to \infty$ and $\operatorname{dom} f \neq \emptyset$. Then there exists $\alpha \in \mathbb{R}_+$ such that $I := (-\infty, -\alpha) \subset \operatorname{dom} f \subset \operatorname{cl} I$, f is strictly convex and increasing on $\operatorname{dom} f$, and $\lim_{x\to-\infty} f(x) = 0 = \inf f$. Moreover,*

$$f'(x) = \sum_{n\geq 1} f_n'(x) = \sum_{n\geq 1} \sigma_n e^{\sigma_n x} \quad \forall x \in \operatorname{int}(\operatorname{dom} f) = I, \tag{6}$$

f' is increasing and continuous on I, $\lim_{x\to-\infty} f'(x) = 0$, and

$$\lim_{x\uparrow -\alpha} f'(x) = \sum_{n\geq 1} \sigma_n e^{-\sigma_n \alpha} =: \gamma \in (0, \infty]. \tag{7}$$

In particular, $\partial f(\operatorname{int}(\operatorname{dom} f)) = f'(I) = (0, \gamma)$.

(iii) *Assume that* $0 < \sigma_n \to \infty$ *is such that* $\operatorname{int}(\operatorname{dom} f) = I := (-\infty, -\alpha)$ *with* $\alpha \in \mathbb{R}_+^*$. *Then either* (a) $\operatorname{dom} f = I$ *and* $\gamma = \infty$, *or* (b) $\operatorname{dom} f = \operatorname{cl} I$ *and* $\gamma = \infty$, *in which case* $f'_-(-\alpha) = \gamma$, $\partial f(-\alpha) = \emptyset$ *and the series* $\sum_{n\geq 1} f'_n(-\alpha)$ *is not convergent, or* (c) $\operatorname{dom} f = \operatorname{cl} I$ *and* $\gamma < \infty$, *in which case* $f'_-(-\alpha) = \gamma$ *and*

$$\partial f(-\alpha) = [\gamma, \infty) \neq \{\gamma\} = \sum_{n\geq 1} \partial f_n(-\alpha).$$

Proof. (i) Take $\overline{x} \in \operatorname{dom} f$. Then the series $\sum_{n\geq 1} e^{\sigma_n \overline{x}}$ is convergent, and so $e^{\sigma_n \overline{x}} \to 0$. The conclusion is obvious.

(ii) Set $\beta := \sup(\operatorname{dom} f) \in (-\infty, \infty]$ and take $x < \beta$. Then there exists $\overline{x} \in \operatorname{dom} f$ with $x \leq \overline{x}$. Because $\sigma_n \geq 0$, and so $0 < e^{\sigma_n x} \leq e^{\sigma_n \overline{x}}$ for $n \geq 1$, the series $\sum_{n\geq 1} e^{\sigma_n x}$ is convergent, whence $x \in \operatorname{dom} f$. Hence $(-\infty, \beta) \subset \operatorname{dom} f \subset (-\infty, \beta]$. Since $0 \notin \operatorname{dom} f$, we obtain that $\beta \leq 0$, and so $\alpha := -\beta$ does the job. Since f_n is strictly convex and increasing, f is strictly convex and increasing on its domain.

Because $f'_n(x) = \sigma_n e^{\sigma_n x}$ for every $x \in \mathbb{R}$, we get (6) using Corollary 11. From (6) we have that f' is increasing and continuous on $\operatorname{int}(\operatorname{dom} f)$.

Since $0 < \sigma_n e^{\sigma_n x} \leq \sigma_n e^{\sigma_n \overline{x}}$ for all $n \geq 1$ and $x \leq \overline{x}$, the series $\sum_{n\geq 1} \sigma_n e^{\sigma_n x}$ is uniformly convergent (u.c. for short) on $(-\infty, \overline{x}]$ for any $\overline{x} \in \operatorname{int}(\operatorname{dom} f)$. Since $\lim_{x\to-\infty} (\sigma_n e^{\sigma_n x}) = \lim_{x\to-\infty} e^{\sigma_n x} = 0$, from (6) and $f = \sum_{n\geq 1} f_n$ we obtain that $\lim_{x\to-\infty} f'(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, respectively.

From (6) we have that

$$f'(x) = \sum_{n\geq 1} \sigma_n e^{\sigma_n x} \geq \sum_{k=1}^{n} \sigma_k e^{\sigma_k x} \quad \forall x \in I,\ \forall n \geq 1. \tag{8}$$

Taking the limit for $-\alpha > x \to -\alpha$, we get $\lim_{x\uparrow-\alpha} f'(x) \geq \sum_{k=1}^{n} \sigma_k e^{-\sigma_k \alpha}$. Taking now the limit for $n \to \infty$ we get $\lim_{x\uparrow-\alpha} f'(x) \geq \gamma := \sum_{n\geq 1} \sigma_n e^{-\sigma_n \alpha} \in (0, \infty]$. If $\gamma = \infty$ it is clear that (7) holds. Assume that $\gamma < \infty$. There exists some $n_0 \geq 1$ such that $\sigma_n \geq 1$, whence $\sigma_n e^{-\sigma_n \alpha} \geq e^{-\sigma_n \alpha}$, for $n \geq n_0$, and so $\sum_{n\geq 1} e^{-\sigma_n \alpha}$ is convergent. It follows that the series $\sum_{n\geq 1} f_n$ and $\sum_{n\geq 1} f'_n$ are u.c. on $(-\infty, -\alpha]$, and so $\lim_{x\uparrow-\alpha} f'(x) = \sum_{n\geq 1} \lim_{x\uparrow-\alpha} f'_n(x) = \sum_{n\geq 1} \sigma_n e^{-\sigma_n \alpha}$, that is (7) holds in this case, too.

(iii) Let $\alpha \in \mathbb{R}_+^*$. Since $0 < e^{-\sigma_n \alpha} \leq \sigma_n e^{-\sigma_n \alpha}$ for large n, we get $\gamma = \infty$ when $-\alpha \notin \operatorname{dom} f$, and so (a) holds. Assume that $-\alpha \in \operatorname{dom} f$. Since $f \in \Gamma(\mathbb{R})$, we have that f'_- is continuous from the left, whence $f'_-(-\alpha) = \lim_{x\uparrow-\alpha} f'(x)$, and $\partial f(-\alpha) = [f'_-(-\alpha), \infty)$. Now the conclusion is immediate using (7). □

Example 13 *In Proposition 12, for* $\sigma_n = n^\theta$ $(n \geq 1)$ *with* $\theta > 0$ *one has* $\operatorname{dom} f = (-\infty, 0)$, *for* $\sigma_n = \ln\left[n(\ln n)^\theta\right]$ $(n \geq 2)$ *with* $\theta \in \mathbb{R}$ *one has* $\operatorname{int}(\operatorname{dom} f) = (-\infty, -1)$, *while for* $\sigma_n = \ln(\ln n)$ $(n \geq 2)$ *one has* $\operatorname{dom} f = \emptyset$. *Moreover, let* $\sigma_n = \ln\left[n(\ln n)^\theta\right]$ $(n \geq 2)$;[1] *for* $\theta \in (-\infty, 1]$ *one has* $\operatorname{dom} f = (-\infty, -1)$, *for* $\theta \in (1, 2]$ *one has* $\operatorname{dom} f = (-\infty, -1]$ *and* $f'_-(-1) = \infty$, *for* $\theta \in (2, \infty)$ *one has* $\operatorname{dom} f = (-\infty, -1]$ *and* $f'_-(-1) < \infty$.

Proposition 12 (iii) (b) and Example 13 show that the conclusion of Theorem 9 can be false for $x \in \operatorname{dom} f \setminus \operatorname{int}(\operatorname{dom} f)$ [even for $x \in \operatorname{dom}(\partial f) \setminus \operatorname{int}(\operatorname{dom} f)$].

One could ask if the condition $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$ in Theorem 9 is just a technical assumption. The next example shows that this condition is essential.

[1]We thank Prof. C. Lefter for this example.

Example 14 *Let $g_n(x,y) := e^{nx+(-1)^n \varsigma_n y}$ for $x, y \in \mathbb{R}$ and $g = \sum_{n\geq 1} g_n$, where $(\varsigma_n)_{n\geq 1} \subset \mathbb{R}$ is such that $\varsigma_n / n \to \infty$. Clearly $g, g_n \in \Gamma(\mathbb{R}^2)$ with $g(x,y) = f(x) + \iota_{\{0\}}(y)$ for $(x,y) \in \mathbb{R}^2$, where*

$$f(x) = \sum_{n\geq 1} e^{nx} = \begin{cases} \frac{e^x}{1-e^x} & \text{if } x \in \mathbb{R}_-^*, \\ \infty & \text{if } x \in \mathbb{R}_+. \end{cases} \tag{9}$$

Hence $\operatorname{dom} g = \mathbb{R}_-^* \times \{0\} = \operatorname{ri}(\operatorname{dom} g)$, *but* $\operatorname{int}(\operatorname{dom} g) = \emptyset$. *It is clear that for $(x,y) \in \operatorname{ri}(\operatorname{dom} g) = \mathbb{R}_-^* \times \{0\}$ we have that $\partial g(x,0) = \partial f(x) \times \partial \iota_{\{0\}}(0) = \{e^x/(1-e^x)^2\} \times \mathbb{R}$. However, $\partial g_n(x,0) = \{\nabla g_n(x,0)\} = \{(ne^{nx}, (-1)^n \varsigma_n e^{nx})\}$. For $\varsigma_n := n^2$ $(n \geq 1)$ we get*

$$\sum_{n\geq 1} \nabla g_n(x,0) = \big(f'(x), 8f''(2x) - f''(x)\big) \quad \forall x \in \mathbb{R}_-^*, \tag{10}$$

for $\varsigma_n = e^{\alpha n}$ with $\alpha > 0$ we get $\sum_{n\geq 1} \nabla g_n(x,0) = \big(f'(x), -e^{x+\alpha}/(1+e^{x+\alpha})\big)$ for $x < -\alpha$ and $\sum_{n\geq 1} \nabla g_n(x,0)$ is not convergent for $x \in [-\alpha, 0)$, while for $\varsigma_n = e^{n^2}$ $(n \geq 1)$ the series $\sum_{n\geq 1} \nabla g_n(x,0)$ is not convergent for each $x \in \mathbb{R}_-^*$.

Indeed, we have that $f'(x) = \sum_{n\geq 1} ne^{nx} = e^x/(1-e^x)^2$ and $f''(x) = \sum_{n\geq 1} n^2 e^{nx}$ for $x \in \mathbb{R}_-^*$. It follows that for $x \in \mathbb{R}_-^*$ we have

$$8f''(2x) = 2\sum_{n\geq 1}(2n)^2 e^{2nx} = \sum_{n\geq 1}(-1)^n n^2 e^{nx} + \sum_{n\geq 1} n^2 e^{nx} = \sum_{n\geq 1}(-1)^n n^2 e^{nx} + f''(x),$$

whence (10) follows for $\varsigma_n = n^2$.

3 Applications to the conjugate of a countable sum

A natural question is what we could say about the conjugate of $f = \sum_{n\geq 1} f_n$ when $f, f_n \in \Lambda(E)$. It is known that for a finite family of functions $f_1, \ldots, f_n \in \Lambda(E)$ one has

$$\begin{aligned}(f_1 + \ldots + f_n)^*(x^*) &\leq (f_1^* \Box \ldots \Box f_n^*)(x^*) \\ &:= \inf\{f_1^*(x_1^*) + \ldots + f_n^*(x_n^*) \mid x_1^* + \ldots + x_n^* = x^*\}.\end{aligned}$$

Of course, in the above formula one could take $x_k^* \in \operatorname{dom} f_k^*$ for every $k \in \overline{1,n}$ using the usual convention $\inf \emptyset := \infty$. Moreover, when all functions (but one) are continuous at some point in $\cap_{k\in\overline{1,n}} \operatorname{dom} f_k$ we have, even for every $x^* \in E^*$,

$$(f_1 + \ldots + f_n)^*(x^*) = \min\{f_1^*(x_1^*) + \ldots + f_n^*(x_n^*) \mid x_1^* + \ldots + x_n^* = x^*\}.$$

Recall that the inf-convolution operation $\Box$ was introduced by J. J. Moreau in [5]; many properties of this operation can be found in [6], among them being the formula mentioned above. The aim of this section is to extend and study this operation to countable sums of convex functions.

In the next proposition we put together several assertions on the conjugate of $\sum_{n\geq 1} f_n$; the last assertion is an application of Theorem 9.

Proposition 15 *Let $f, f_n \in \Lambda(E)$. Assume that $f(x) = \sum_{n\geq 1} f_n(x)$ for every $x \in E$.*

(i) *If* $(x_n^*)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1}$ *is such that* $w^*\text{-}\sum_{n\geq 1} x_n^* = x^* \in E^*$, *then the sequence* $\sum_{k=1}^n f_k^*(x_k^*)$ *has a limit in* $(-\infty, \infty]$ *and* $f^*(x^*) \leq \sum_{n\geq 1} f_n^*(x_n^*)$; *in particular,*

$$f^*(x^*) \leq \inf \left\{ \sum_{n\geq 1} f_n^*(x_n^*) \mid (x_n^*)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1},\ x^* = w^*\text{-}\sum_{n\geq 1} x_n^* \right\} \tag{11}$$

for every $x^* \in E^*$.

(ii) *If there exists* $(x_n^*)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1}$ *such that* $w^*\text{-}\sum_{n\geq 1} x_n^* = x^* \in E^*$ *and* $\sum_{n\geq 1} f_n^*(x_n^*) \in \mathbb{R}$ *then* $x^* \in \operatorname{dom} f^*$.

(iii) *If* $\overline{x} \in \cap_{n\geq 1} \operatorname{dom} f_n$ *and* $(\overline{x}_n^*)_{n\geq 1} \subset (\partial f_n(\overline{x}))_{n\geq 1}$ *is such that* $w^*\text{-}\sum_{n\geq 1} \overline{x}_n^* = x^* \in E^*$, *then* $\overline{x} \in \operatorname{dom} f$, $x^* \in \partial f(\overline{x})$ *and* $f^*(x^*) = \sum_{n\geq 1} f_n^*(\overline{x}_n^*)$. *In particular,*

$$f^*(x^*) = \min \left\{ \sum_{n\geq 1} f_n^*(x_n^*) \mid (x_n^*)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1},\ x^* = w^*\text{-}\sum_{n\geq 1} x_n^* \right\} \tag{12}$$

(iv) *Let* $x^* \in \partial f(\overline{x})$ *with* $\overline{x} \in \operatorname{dom} f$ *and let* $(x_n^*)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1}$ *be such that* $x^* = w^*\text{-}\sum_{n\geq 1} x_n^*$. *Then* $f^*(x^*) = \sum_{n\geq 1} f_n^*(x_n^*)$ *if and only if* $x_n^* \in \partial f_n(\overline{x})$ *for every* $n \geq 1$.

(v) *Assume that* f *and* f_n $(n \geq 1)$ *are continuous on* $\operatorname{int}(\operatorname{dom} f)$. *Then for every* $x^* \in \partial f(\operatorname{int}(\operatorname{dom} f))$ *relation (12) holds. More precisely, if* $x^* \in \partial f(x)$ *for* $x \in \operatorname{int}(\operatorname{dom} f)$ *then* $x^* = w^*\text{-}\sum_{n\geq 1} x_n^*$ *for some* $(x_n^*)_{n\geq 1} \subset (\partial f_n(x))_{n\geq 1}$, *and* $f^*(x^*) = \sum_{n\geq 1} f_n^*(x_n^*)$.

Proof. (i) Let us fix $(x_n^*)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1}$ with $w^*\text{-}\sum_{n\geq 1} x_n^* = x^* \in X^*$; take $x \in \operatorname{dom} f$ and $\gamma_n := f_n(x) + f_n^*(x_n^*) - \langle x, x_n^* \rangle \geq 0$ for $n \geq 1$. Hence $\sum_{k=1}^n \gamma_k \to \gamma \in [0, \infty]$, and so

$$\begin{aligned}\sum_{n\geq 1} f_n^*(x_n^*) &= \lim_{n\to\infty} \sum_{k=1}^n f_k^*(x_k^*) = \lim_{n\to\infty} \left(\sum_{k=1}^n \gamma_k - \sum_{k=1}^n f_k(x) + \left\langle x, \sum_{k=1}^n x_k^* \right\rangle \right) \\ &= \gamma - f(x) + \langle x, x^* \rangle \geq \langle x, x^* \rangle - f(x).\end{aligned}$$

It follows that $\sum_{n\geq 1} f_n^*(x_n^*) \geq \sup_{x\in \operatorname{dom} f} (\langle x, x^* \rangle - f(x)) = f^*(x^*) > -\infty$. Relation (11) is now obvious.

(ii) The assertion is an immediate consequence of (11).

(iii) Since $\overline{x}_n^* \in \partial f_n(\overline{x}) \subset \operatorname{dom} f_n^*$, we have that $f_n^*(\overline{x}_n^*) = \langle \overline{x}, \overline{x}_n^* \rangle - f_n(\overline{x})$ for $n \geq 1$, and so, using (i), we get

$$-\infty < f^*(x^*) \leq \sum_{n\geq 1} f_n^*(\overline{x}_n^*) = \sum_{n\geq 1} [\langle \overline{x}, \overline{x}_n^* \rangle - f_n(\overline{x})] = \langle \overline{x}, x^* \rangle - f(\overline{x}) < \infty.$$

It follows that $\overline{x} \in \operatorname{dom} f$ and $f^*(x^*) + f(\overline{x}) \leq \langle \overline{x}, x^* \rangle$, whence $x^* \in \partial f(\overline{x})$ and $f^*(x^*) = \sum_{n\geq 1} f_n^*(\overline{x}_n^*)$. Using again (i) we obtain that (12) holds.

(iv) Since $\operatorname{dom} f \subset \cap_{n\geq 1} \operatorname{dom} f_n$, we have that $\overline{x} \in \cap_{n\geq 1} \operatorname{dom} f_n$. Assuming that $x_n^* \in \partial f_n(\overline{x})$ for $n \geq 1$, the conclusion $f^*(x^*) = \sum_{n\geq 1} f_n^*(x_n^*)$ follows from (iii).

Conversely, assume that $f^*(x^*) = \sum_{n\geq 1} f_n^*(x_n^*)$. Then

$$0 = \sum_{n\geq 1} f_n^*(x_n^*) + \sum_{n\geq 1} f_n(\overline{x}) - \sum_{n\geq 1} \langle \overline{x}, x_n^* \rangle = \sum_{n\geq 1} [f_n^*(x_n^*) + f_n(\overline{x}) - \langle \overline{x}, x_n^* \rangle].$$

Since $f_n^*(x_n^*) + f_n(\overline{x}) - \langle \overline{x}, x_n^* \rangle \geq 0$ for $n \geq 1$, we obtain that $f_n^*(x_n^*) + f_n(\overline{x}) - \langle \overline{x}, x_n^* \rangle = 0$, and so $x_n^* \in \partial f_n(\overline{x})$ for every $n \geq 1$.

(v) Assume now that f and f_n are continuous on $\operatorname{int}(\operatorname{dom} f)$ and take $x^* \in \partial f(\operatorname{int}(\operatorname{dom} f))$. Then there exists $\overline{x} \in \operatorname{int}(\operatorname{dom} f)$ such that $x^* \in \partial f(\overline{x})$. By Theorem 9, there exists $(x_n^*)_{n\geq 1} \subset (\partial f_n(\overline{x}))_{n\geq 1}$ such that $x^* = w^*\text{-}\sum_{n\geq 1} x_n^*$. By (iii) we have that $f^*(x^*) = \sum_{n\geq 1} f_n^*(x_n^*)$. □

Remark 16 *Note that if each function f_n^* (with $n \geq 1$) is strictly convex on its domain, then the infimum in (11), when finite, is attained at at most one sequence $(x_n^*)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1}$ with $x^* = w^*\text{-}\sum_{n\geq 1} x_n^*$.*

Because (12) is valid automatically for $x^* \in E^* \setminus \operatorname{dom} f^*$, the problem is to see what is happening for $x^* \in \operatorname{dom} f^* \setminus \partial f(\operatorname{int}(\operatorname{dom} f))$.

Taking $f_k = 0$ for $k \geq n+1$ in Proposition 15 (v), its conclusion is much weaker than the usual result mentioned at the beginning of this section because nothing is said for sure for $x^* \in \operatorname{dom} f^* \setminus \partial f(\operatorname{int}(\operatorname{dom} f))$.

In the next two propositions we give complete descriptions for f^* and g^*, where f and g are provided in Proposition 12 and Example 14, respectively.

Proposition 17 *Let $f_n(x) := e^{\sigma_n x}$ for $x \in \mathbb{R}$ with $0 < \sigma_n \to \infty$, and $f = \sum_{n\geq 1} f_n$. Assume that $\operatorname{dom} f \neq \emptyset$, and so $I := (-\infty, -\alpha) \subset \operatorname{dom} f \subset \operatorname{cl} I$ for some $\alpha \in \mathbb{R}_+$.*

(i) *Then $\partial f(\operatorname{int}(\operatorname{dom} f)) = (0, \gamma)$, where $\gamma := \sum_{n\geq 1} \sigma_n e^{-\sigma_n \alpha} \in \overline{\mathbb{R}}_+$, $\operatorname{dom} f^* = \mathbb{R}_+$, and*

$$f^*(u) \leq \inf \left\{ \sum_{n\geq 1} f_n^*(u_n) \mid (u_n)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1},\ u = \sum_{n\geq 1} u_n \right\} < \infty \quad \forall u \in \mathbb{R}_+. \tag{13}$$

(ii) *Let $u \in \mathbb{R}_+$ $(= \operatorname{dom} f^*)$. Then $u \in [0, \gamma] \cap \mathbb{R}$ if and only if*

$$f^*(u) = \min \left\{ \sum_{n\geq 1} f_n^*(u_n) \mid (u_n)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1},\ u = \sum_{n\geq 1} u_n \right\}, \tag{14}$$

or, equivalently,

$$f^*(u) = \min \left\{ \sum_{n\geq 1} u_n(\ln u_n - 1) \mid (u_n)_{n\geq 1} \subset \mathbb{R}_+,\ u = \sum_{n\geq 1} \sigma_n u_n \right\}. \tag{15}$$

More precisely, the minimum in (15) is attained for $u_n = 0$ $(n \geq 1)$ when $u = 0$, for $u_n = e^{\sigma_n x}$ $(n \geq 1)$ when $u = f'(x)$ with $x \in I$, for $u_n = e^{-\sigma_n \alpha}$ when $u = \gamma$ $(< \infty)$ (in which case $\alpha \in \mathbb{R}_+^*$, $-\alpha \in \operatorname{dom} \partial f = \operatorname{dom} f$ and $f'_-(-\alpha) = \gamma$).

(iii) *Let $u \in \mathbb{R}_+$ $(= \operatorname{dom} f^*)$. Then*

$$f^*(u) = \inf \left\{ \sum_{n\geq 1} f_n^*(u_n) \mid (u_n)_{n\geq 1} \subset (\operatorname{dom} f_n^*)_{n\geq 1},\ u = \sum_{n\geq 1} u_n \right\} \tag{16}$$

$$= \inf \left\{ \sum_{n\geq 1} u_n(\ln u_n - 1) \mid (u_n)_{n\geq 1} \subset \mathbb{R}_+,\ u = \sum_{n\geq 1} \sigma_n u_n \right\}. \tag{17}$$

Proof. Let $\alpha \in \mathbb{R}_+$ be such that $I := (-\infty, -\alpha) = \operatorname{int}(\operatorname{dom} f) \subset \operatorname{dom} f \subset \operatorname{cl} I$ (see Proposition 12), and take $\gamma := \sum_{n\geq 1} \sigma_n e^{-\sigma_n \alpha} \in (0, \infty]$.

(i) The equality $\partial f(\operatorname{int}(\operatorname{dom} f)) = (0, \gamma)$ is proved in Proposition 12 (ii). Because the conjugate of the exponential function is given by

$$\exp^*(u) = \begin{cases} \infty & \text{if } u \in \mathbb{R}_-^*, \\ u(\ln u - 1) & \text{if } u \in \mathbb{R}_+, \end{cases} \tag{18}$$

where $0 \cdot \ln 0 := 0$, we have $f_n^*(u) = \frac{u}{\sigma_n}(\ln \frac{u}{\sigma_n} - 1)$ for $u \in \mathbb{R}_+$ and $f_n^*(u) = \infty$ for $u \in \mathbb{R}_-^*$.

The first inequality in (13) is given in (11), while for the second inequality just take $u_1 := u \in \mathbb{R}_+$ and $u_n := 0$ for $n \geq 2$. For $u \in \mathbb{R}^*_-$ we have that $f^*(u) = \sup_{x\in\mathbb{R}}[xu - f(x)] \geq \lim_{x\to-\infty}[xu - f(x)] = \infty$. It follows that $\operatorname{dom} f^* = \mathbb{R}_+$.

(ii) Consider $u \in \mathbb{R}_+$. Clearly, $f^*(0) = -\inf f = 0$, and so (14) and (15) hold in this case, with attainment for $u_n = 0$ $(n \geq 1)$. For $u \in (0, \gamma) = \partial f(\operatorname{int}(\operatorname{dom} f)) = f'(I)$, there exists $x \in I$ such that $f'(x) = u$, and so

$$f^*(u) = f^*(f'(x)) = \sup_{x\in\mathbb{R}}[xu - f(x)] = xf'(x) - f(x) = \sum_{n\geq1} e^{\sigma_n x}(\sigma_n x - 1) \tag{19}$$

$$= \sum_{n\geq1} f_n^*(f_n'(x)) = \min\left\{\sum_{n\geq1} f_n^*(u_n) \mid (u_n)_{n\geq1} \subset \mathbb{R}_+,\ u = \sum_{n\geq1} u_n\right\}, \tag{20}$$

the last two equalities being given by Proposition 15 (v). Hence (14) and (15) hold in this case, too, the attainment in (15) being for $u_n = e^{\sigma_n x}$ $(n \geq 1)$. Hence, if $\gamma = \infty$ we have that (14) (therefore, also (15)) holds for all $u \in \mathbb{R}_+ = \operatorname{dom} f^* = [0, \gamma] \cap \mathbb{R}$.

Assume that $\gamma < \infty$; then, by Proposition 12, we have that $\alpha \in \mathbb{R}^*_+$ and $\gamma = f'_-(-\alpha)$. Take $u \geq \gamma$. Since $\psi'(x) = u - f'(x) > 0$ for every $x \in I$, where $\psi(x) := xu - f(x)$, it follows that ψ is increasing on $(-\infty, -\alpha]$, and so

$$f^*(u) = \sup_{x\in(-\infty,-\alpha]} \psi(x) = \psi(-\alpha) = -\alpha u - f(-\alpha). \tag{21}$$

Since f^* is continuous on $\mathbb{R}^*_+ = \operatorname{int}(\operatorname{dom} f^*)$ and $\lim_{x\uparrow-\alpha} f'(x) = \gamma$, taking the limit for $x \uparrow -\alpha$ in (19), we get

$$f^*(\gamma) = -\alpha\gamma - f(-\alpha) = -\alpha \sum_{n\geq1} \sigma_n e^{-\sigma_n \alpha} - \sum_{n\geq1} e^{-\sigma_n\alpha} = \sum_{n\geq1} f_n^*(f_n'(-\alpha)).$$

Assume now that (14) holds for some $u > \gamma$, that is there exists $(u_n)_{n\geq1} \subset \mathbb{R}_+$ such that $u = \sum_{n\geq1} u_n$ and $f^*(u) = \sum_{n\geq1} f_n^*(u_n)$. Because $u \in \partial f(-\alpha) = [\gamma, \infty)$, by Proposition 12 (iv), we obtain that $u_n \in \partial f_n(-\alpha) = \{\sigma_n e^{-\sigma_n\alpha}\}$, whence $u = \sum_{n\geq1} \sigma_n e^{-\sigma_n\alpha} = \gamma$. This contradiction proves that (14) holds if and only if $u \in [0, \gamma] \cap \mathbb{R}$.

(iii) Because by (ii) the conclusion is clearly true for $u \in [0, \gamma] \cap \mathbb{R}$, we may (and do) assume that $\gamma < \infty$. Take $u > \gamma$ and denote by $F(u)$ the (real) number in the RHS of (17). There exists $\overline{n} \geq 1$ such that $\sigma_n \geq u$ for $n \geq \overline{n}$, and fix $n \geq \overline{n}$. For $q \in \mathbb{N}^*$, since $v := u - \gamma + \sum_{k\geq n+1} \sigma_k e^{-\sigma_k\alpha} \in (0, u)$, there exists a unique $\lambda_q \in \mathbb{R}^*_+$ such that $\sum_{k=n+1}^{n+q} \sigma_k e^{-\sigma_k\lambda_q} = v$. It follows that $\lambda_q < \lambda_{q+1} < \alpha$ for all $q \geq 1$; this follows easily by contradiction. Therefore, $\lambda_q \uparrow \mu$ with $\mu \leq \alpha$. Setting $u_k^q := e^{-\sigma_k\alpha}$ for $k \in \overline{1, n}$, $u_k^q := e^{-\sigma_k\lambda_q}$ for $k \in \overline{n+1, n+q}$ and $u_k^q := 0$ for $k > n + q$, we have that $\sum_{k\geq1} \sigma_k u_k^q = u$; taking into account (21), we get

$$f^*(u) \leq F(u) \leq \sum_{k\geq1} u_k^q(\ln u_k^q - 1) = \sum_{k=1}^{n} e^{-\sigma_k\alpha}(-\sigma_k\alpha - 1) + \sum_{k=n+1}^{n+q} e^{-\sigma_k\lambda_q}(-\sigma_k\lambda_q - 1)$$

$$= (\lambda_q - \alpha)\sum_{k=1}^{n} \sigma_k e^{-\sigma_k\alpha} - \lambda_q u - \sum_{k=1}^{n} e^{-\sigma_k\alpha} - \sum_{k=n+1}^{n+q} e^{-\sigma_k\lambda_q} = f^*(u) + \Lambda_q^n,$$

where

$$\Lambda_q^n := (\alpha - \lambda_q)\Big(u - \sum_{k=1}^{n} \sigma_k e^{-\sigma_k \alpha}\Big) + \sum_{k\geq n+1} e^{-\sigma_k \alpha} - \sum_{k=n+1}^{n+q} e^{-\sigma_k \lambda_q}; \tag{22}$$

hence $\Lambda_q^n \geq 0$ for all $n \geq \overline{n}$ and $q \geq 1$.

Assume that $\mu < \alpha$. Since $\lambda_q < \mu$, we have that $\sum_{k=n+1}^{n+q} e^{-\sigma_k \lambda_q} \geq \sum_{k=n+1}^{n+q} e^{-\sigma_k \mu} \to \infty$ for $q \to \infty$. From (22) we get the contradiction $0 \leq \lim_{q\to\infty} \Lambda_q^n = -\infty$. Hence $\mu = \alpha$. Using again (22) we obtain that $\limsup_{q\to\infty} \Lambda_q^n \leq \sum_{k\geq n+1} e^{-\sigma_k \alpha}$, and so $F(u) \leq f^*(u) + \sum_{k\geq n+1} e^{-\sigma_k \alpha}$ for every $n \geq \overline{n}$. It follows that $F(u) \leq f^*(u)$. Therefore, $F(u) = f^*(u)$, and so (17) holds. The proof is complete. □

Observe that the condition $\sigma_n > 0$ in Proposition 17 is not essential; because $\sigma_n \to \infty$, $\sigma_n > 0$ for some $n_0 \geq 1$ and every $n \geq n_0$. Indeed, apply Proposition 17 for $g := \sum_{n\geq n_0} f_n$, then the usual duality results for $f = f_1 + \ldots + f_{n_0-1} + g$. Of course, in the conclusion one must replace $[0,\gamma] \cap \mathbb{R}$ by $\operatorname{cl}[\partial f(\operatorname{int}(\operatorname{dom} f))]$.

One could ask, as for Theorem 9, if the condition $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$ is essential in Proposition 15 (v). The next result proves that this is the case.

Proposition 18 *Let g, g_n be as in Example 14, where $(\varsigma_n)_{n\geq 1} \subset \mathbb{R}_+$ is such that $\varsigma_n/n \to \infty$. Then $\operatorname{dom} g^* = \mathbb{R}_+ \times \mathbb{R}$ and*

$$g^*(u,v) = \inf\Big\{\sum_{n\geq 1} \gamma_n(\ln\gamma_n - 1) \mid \gamma_n \geq 0,\ u = \sum_{n\geq 1} n\gamma_n,\ v = \sum_{n\geq 1} (-1)^n \varsigma_n \gamma_n\Big\} \tag{23}$$

for all $(u,v) \in \big(\mathbb{R}_+^* \times \mathbb{R}\big) \cup \{(0,0)\}$, while for $u = 0 \neq v$ the term in the RHS of (23) is ∞. Moreover, for $u = 0 = v$ the infimum in (23) is attained, while for $u \in \mathbb{R}_+^*$ the infimum in (23) is attained if and only if $\overline{v} := \sum_{n\geq 1} (-1)^n \varsigma_n \Big(\frac{1+2u-\sqrt{4u+1}}{2u}\Big)^n \in \mathbb{R}$ and $v = \overline{v}$.

Proof. On one hand, because $g(x,y) = f(x) + \iota_{\{0\}}(y)$, where f is defined in (9), we have that $g^*(u,v) = f^*(u)$ for $(u,v) \in \mathbb{R}^2$. On the other hand $g_n^*(u,v) = \frac{u}{n}(\ln\frac{u}{n} - 1)$ for $u \in \mathbb{R}_+$ and $v = (-1)^n \varsigma_n/n$, while $g_n^*(u,v) = \infty$ otherwise. By Proposition 15 (i) we have that

$$g^*(u,v) \leq G(u,v) := \inf\Big\{\sum_{n\geq 1} \gamma_n(\ln\gamma_n - 1) \mid \gamma_n \geq 0,\ u = \sum_{n\geq 1} n\gamma_n,\ v = \sum_{n\geq 1} (-1)^n \varsigma_n \gamma_n\Big\} \tag{24}$$

for all $(u,v) \in \mathbb{R}_+ \times \mathbb{R}$.

It is clear that for $u = v = 0$ one has equality with attained infimum (for $\gamma_n = 0$ for every $n \geq 1$), while for $u = 0 \neq v$ the RHS term of (24) is ∞.

Applying Proposition 17 for $\sigma_n := n$ ($n \geq 1$) we have that $\alpha = 0$ and $\gamma = \infty$; moreover, for $u \in \mathbb{R}_+^*$,

$$f^*(u) = \min\Big\{\sum_{n\geq 1} \gamma_n(\ln\gamma_n - 1) \mid \gamma_n \geq 0,\ u = \sum_{n\geq 1} n\gamma_n\Big\},$$

which is attained only for the sequence $(\overline{\gamma}_n^u)_{n\geq 1}$, where

$$\overline{\gamma}_n^u := \left(\frac{1+2u-\sqrt{4u+1}}{2u}\right)^n \quad (n \geq 1). \tag{25}$$

Consequently, we have equality in (24) with attained infimum if and only if $\overline{v} \in \mathbb{R}$ and $v = \overline{v}$.

Let $(u,v)\in\mathbb{R}_+^*\times\mathbb{R}$ and fix $\varepsilon>0$. Then there exists some $\overline{n}\geq 1$ such that $\sum_{k=1}^{\overline{n}}\bar{\gamma}_k^u(\ln\bar{\gamma}_k^u-1)<f^*(u)+\varepsilon/2$. Set $\overline{u}:=\sum_{k=1}^{\overline{n}}k\bar{\gamma}_k^u$ and $\overline{v}:=\sum_{k=1}^{\overline{n}}(-1)^k\varsigma_k\bar{\gamma}_k^u$; then $u':=u-\overline{u}>0$ and $v':=v-\overline{v}\in\mathbb{R}$. Observe that for a fixed $n\in\mathbb{N}^*$, since g_n is finite (and continuous) one has $(g_n+g_{n+1})^*=g_n^*\Box g_{n+1}^*$ with exact convolution, and so

$$\operatorname{dom}(g_n+g_{n+1})^*=\operatorname{dom}g_n^*+\operatorname{dom}g_{n+1}^*=\mathbb{R}_+(n,(-1)^n\varsigma_n)+\mathbb{R}_+(n+1,(-1)^{n+1}\varsigma_{n+1}).$$

Take $\overline{n}\geq 1$ such that $u'\varsigma_n\geq n\,|v'|$ for $n\geq\overline{n}$. Then

$$\gamma_n':=\frac{u'\varsigma_{n+1}-(-1)^{n+1}(n+1)v'}{n\varsigma_{n+1}+(n+1)\varsigma_n}\geq 0,\quad \gamma_{n+1}':=\frac{u'\varsigma_n-(-1)^n nv'}{n\varsigma_{n+1}+(n+1)\varsigma_n}\geq 0,$$

and $(u',v')=\gamma_n'(n,(-1)^n\varsigma_n)+\gamma_{n+1}'(n+1,(-1)^{n+1}\varsigma_{n+1})\in\operatorname{dom}(g_n+g_{n+1})^*$. Moreover,

$$\begin{aligned}(g_n+g_{n+1})^*(u',v')&=g_n^*(n\gamma_n',(-1)^n\varsigma_n\gamma_n')+g_{n+1}^*((n+1)\gamma_{n+1}',(-1)^{n+1}\varsigma_{n+1}\gamma_{n+1}')\\&=\gamma_n'(\ln\gamma_n'-1)+\gamma_{n+1}'(\ln\gamma_{n+1}'-1).\end{aligned}$$

Since for $n\geq\overline{n}$ one has $0\leq\gamma_n'\leq\frac{u'\varsigma_{n+1}+(n+1)|v'|}{n\varsigma_{n+1}}\leq u'\frac{1}{n}+|v'|\frac{2}{\varsigma_{n+1}}\to 0$ and $0\leq\gamma_{n+1}'\leq\frac{u'\varsigma_n+n|v'|}{(n+1)\varsigma_n}=u'\frac{1}{n+1}+|v'|\frac{1}{\varsigma_n}\to 0$, there exists $m\geq\overline{n}$ such that $\gamma_m'(\ln\gamma_m'-1)+\gamma_{m+1}'(\ln\gamma_{m+1}'-1)<\varepsilon/2$; set $\gamma_k:=\bar{\gamma}_k^u$ for $k\in\overline{1,\overline{n}}$, $\gamma_k:=\gamma_k'$ for $k\in\{m,m+1\}$ and $\gamma_k:=0$ otherwise. Then $\gamma_n\geq 0$ for all $n\geq 1$, $u=\sum_{n\geq 1}n\gamma_n$ and $v=\sum_{n\geq 1}(-1)^n\varsigma_n\gamma_n$; moreover,

$$\begin{aligned}G(u,v)&\leq\sum_{n\geq 1}\gamma_n(\ln\gamma_n-1)=\sum_{k=1}^{\overline{n}}\bar{\gamma}_k^u(\ln\bar{\gamma}_k^u-1)+\gamma_m'(\ln\gamma_m'-1)+\gamma_{m+1}'(\ln\gamma_{m+1}'-1)\\&\leq f^*(u)+\varepsilon/2+\varepsilon/2=f^*(u)+\varepsilon=g^*(u,v)+\varepsilon.\end{aligned}$$

It follows that $G(u,v)\leq g^*(u,v)$, and so (23) holds. The proof is complete. □

Remark 19 *Observe that, depending on $(\varsigma_n)_{n\geq 1}$, the set of those $u>0$ for which the infimum in the RHS of (23) is attained for some $v\in\mathbb{R}$ could be $\mathbb{R}_+^*$, a proper subset of $\mathbb{R}_+^*$, or the empty set. For example, when $\varsigma_n=n^k$ with $k>1$ the series $\sum_{n\geq 1}(-1)^n\varsigma_n\overline{\gamma}_n^u$ is convergent. If $\varsigma_n=e^{n\alpha}$ with $\alpha>0$ the series $\sum_{n\geq 1}(-1)^n\varsigma_n\overline{\gamma}_n^u$ is convergent iff $\alpha<\ln\frac{1+2u+\sqrt{4u+1}}{2u}$. If $\varsigma_n=e^{n^2}$, the series $\sum_{n\geq 1}(-1)^n\varsigma_n\overline{\gamma}_n^u$ is not convergent. (Here $\overline{\gamma}_n^u$ is defined in (25).)*

Having in view Propositions 17 and 18, the following question is natural:

Question 2 *Let $f,f_n\in\Lambda(\mathbb{R}^p)$ with $p\in\mathbb{N}^*$ and $f(x)=\sum_{n\geq 1}f_n(x)$ for every $x\in\mathbb{R}^p$; is it true that*

$$f^*(u)=\inf\left\{\sum_{n\geq 1}f_n^*(u_n)\mid(u_n)_{n\geq 1}\subset(\operatorname{dom}f_n^*)_{n\geq 1},\ u=\sum_{n\geq 1}u_n\right\}$$

for all $u\in\operatorname{int}(\operatorname{dom}f^*)$?

M. Valadier in [8] defines the continuous inf-convolution for a family $(f_t)_{t\in T}$ of proper lower semicontinuous functions defined on $\mathbb{R}^p$, where $(T,\mathcal{T},\mu)$ is a measure space with $\mu\geq 0$ being σ-finite. In the case in which $T:=\mathbb{N}^*$, $\mathcal{T}:=2^{\mathbb{N}}$ and $\mu:\mathcal{T}\to[0,\infty]$ is defined by $\mu(A):=\operatorname{card}A$, [8, Th. 7] has the following (equivalent) statement:

Theorem *Let $(g_n)_{n\geq 1} \subset \Gamma(\mathbb{R}^p)$ be such that the series $\sum_{n\geq 1} |g_n^*(u)| < \infty$ for every $u \in \mathbb{R}^p$. Then the function $\widetilde{g} : \mathbb{R}^p \to \overline{\mathbb{R}}$, defined by*

$$\widetilde{g}(x) := \inf \left\{ \sum_{n\geq 1} g_n(x_n) \mid (x_n)_{n\geq 1} \subset \mathbb{R}^p,\ \sum_{n\geq 1} \|x_n\| < \infty,\ x = \sum_{n\geq 1} x_n \right\},$$

belongs to $\Gamma(\mathbb{R}^p)$, the infimum above is attained for every $x \in \mathbb{R}^p$, and $\widetilde{g}^* = \sum_{n\geq 1} g_n^*$.

Taking $(f_n)_{n\geq 1} \subset \Gamma(\mathbb{R}^p)$, and setting $g_n := f_n^*$ (hence $g_n^* = f_n$), the hypothesis of [8, Th. 7] implies that $\operatorname{dom} f = \mathbb{R}^p$ (and of course f, f_n are continuous on $\mathbb{R}^p$), an hypothesis which is stronger than that of Proposition 15 (v), but also the conclusion of [8, Th. 7] is stronger. Clearly, [8, Th. 7] (above) can not be applied in the previous examples (because f is not finite-valued), as well as for the example in the next section.[2]

4 An application to entropy minimization

As mentioned in Introduction, in Statistical Physics (Statistical Mechanics) one has to minimize $\sum_{i\in I} n_i(\ln n_i - 1)$ with the constraints $\sum_{i\in I} n_i = N$ and $\sum_{i\in I} n_i\varepsilon_i = \varepsilon$, where I is a countable set and n_i are nonnegative integers. In this context consider $h_i := \exp \circ A_i$, where $A_i : \mathbb{R}^2 \to \mathbb{R}$ is defined by $A_i(x, y) := x + \varepsilon_i y$, and $h := \sum_{i\in I} h_i$. Because $h_i > 0$, we have that $\sum_{i\in I} h_i = \sum_{j\in J} h_{p(j)}$ for every bijection $p : J \to I$. So, we (can) take $I = \mathbb{N}^*$, the set of positive integers, $(\sigma_n)_{n\geq 1} \subset \mathbb{R}$, $A_n(x, y) := x + \sigma_n y$, $h_n := \exp \circ A_n$ and $h := \sum_{n\geq 1} h_n$. Hence

$$h(x, y) = \sum_{n\geq 1} e^{x+\sigma_n y} = e^x \sum_{n\geq 1} e^{\sigma_n y} = e^x f(y)$$

with $f : \mathbb{R} \to \overline{\mathbb{R}}$, $f(y) := \sum_{n\geq 1} e^{\sigma_n y}$. By Proposition 12, when $\operatorname{dom} h = \mathbb{R} \times \operatorname{dom} f \neq \emptyset$, we may (and do) assume that $\sigma_n \to \infty$, in which case there exists $\alpha \in \mathbb{R}_+$ such that $I := (-\infty, -\alpha) = \operatorname{int}(\operatorname{dom} f) \subset \operatorname{dom} f \subset \operatorname{cl} I$. Since h_n is differentiable on $\mathbb{R}^2$ [with $\nabla h_n(x, y) = e^{x+\sigma_n y}(1, \sigma_n)$], by Theorem 9 (and Corollary 11), h is differentiable on $\operatorname{int}(\operatorname{dom} h) = \mathbb{R} \times I$ and

$$\nabla h(x, y) = \sum_{n\geq 1} e^{x+\sigma_n y}(1, \sigma_n) = \left(\sum_{n\geq 1} e^{x+\sigma_n y}, \sum_{n\geq 1} \sigma_n e^{x+\sigma_n y} \right) = \big(e^x f(y), e^x f'(y)\big)$$

for all $(x, y) \in \mathbb{R} \times I$. Moreover, because $\operatorname{Im} A_n = \mathbb{R}$ and $A_n^* w = w(1, \sigma_n)$ for $w \in \mathbb{R}$,

$$h_n^*(u, v) = \min \{\exp^*(w) \mid A_n^* w = (u, v)\} = \begin{cases} u(\ln u - 1) & \text{if } u \geq 0 \text{ and } v = u\sigma_n, \\ \infty & \text{otherwise.} \end{cases}$$

Using Proposition 15 (v), it follows that for $(u, v) = \nabla f(x, y)$ with $(x, y) \in \operatorname{int}(\operatorname{dom} f)$,

$$h^*(u, v) = \min \left\{ \sum_{n\geq 1} u_n(\ln u_n - 1) \mid (u_n)_{n\geq 1} \in \mathbb{R}_+,\ \sum_{n\geq 1} u_n = u,\ \sum_{n\geq 1} u_n\sigma_n = v \right\} \tag{26}$$

for every $(u, v) \in \partial h\,(\operatorname{int}(\operatorname{dom} f))$; moreover, because h_n^* is strictly convex, for $(u, v) \in \partial h(x, y)$ with $(x, y) \in \operatorname{int}(\operatorname{dom} h)$, the minimum in (26) is realized at the unique sequence $(\overline{u}_n)_{n\geq 1} = (e^{x+\sigma_n y})_{n\geq 1}$.

[2] We thank Prof. L. Thibault for bringing to our attention the reference [8].

We apply the preceding considerations for the following example taken from [7, p. 10]:

" $\varepsilon(n_x,n_y,n_z)=\frac{h^2}{8mL^2}(n_x^2+n_y^2+n_z^2)$; $n_x,n_y,n_z=1,2,3,\ldots$ (5)
where h is Planck's constant and m the mass of the particle," and L is the side of a cubical box.

Hence, $\sigma_{k,l,m}:=\kappa(k^2+l^2+m^2)$ with $k,l,m\in\mathbb{N}^*$. We take $\kappa=1$ to (slightly) simplify the calculation. It follows that

$$h(x,y)=\sum_{k,l,m\geq 1}e^{x+y(k^2+l^2+m^2)}=e^x\sum_{k,l,m\geq 1}e^{yk^2}e^{yl^2}e^{ym^2}=e^x\Bigg(\sum_{k\geq 1}e^{yk^2}\Bigg)^3.$$

Clearly, $\operatorname{dom}h=\mathbb{R}\times\mathbb{R}_-^*=\operatorname{int}(\operatorname{dom}h)$. Let us consider

$$f:\mathbb{R}\to\overline{\mathbb{R}},\quad f(y):=\sum\nolimits_{n\geq 1}e^{n^2y};$$

hence $h(x,y)=e^x\,[f(y)]^3$. As observed in Example 13, $I:=\operatorname{dom}f=\mathbb{R}_-^*$, and so $\operatorname{dom}h=\mathbb{R}\times I$. The series $\sum_{k\geq 1}e^{yk^2}$, as well as the series $\sum_{k\geq 1}(k^2)^pe^{yk^2}$ with $p\in\mathbb{N}^*$, are uniformly convergent on the interval $(-\infty,-\gamma]$ for every $\gamma>0$ (because $0\leq e^{yk^2}\leq e^{-\gamma k^2}$ for every $y\in(-\infty,-\gamma]$). It follows that $f^{(p)}(y)=\sum_{k\geq 0}(k^2)^pe^{yk^2}$ for every $p\in\mathbb{N}$ (with $f^{(0)}:=f$, $f^{(1)}:=f'$ ...) and $\lim_{y\to-\infty}f^{(p)}(y)=0$ for $p\in\mathbb{N}$. Moreover, $\lim_{y\to 0-}f(y)=\infty$. This is because $f(y)\geq\sum_{k=1}^n e^{yk^2}$ for every $n\geq 1$ and $\lim_{y\to 0-}\sum_{k=1}^n e^{yk^2}=n$. Hence $h|_{\operatorname{dom}h}\in C^\infty(\operatorname{dom}h)$. We know that h is strictly convex on its domain (as the sum of a series of strictly convex functions). In fact $\ln f$ (with $\ln\infty:=\infty$) is proper, convex and lsc; $\ln f$ is even strictly convex on $\operatorname{dom}f=I$. Indeed, $(\ln f)'=f'/f>0$ and $(\ln f)''=(f''f-(f')^2)/f^2>0$ on I; just use Schwarz inequality in ℓ^2. Moreover, $\lim_{y\to-\infty}f'(y)/f(y)=1$ and $\eta:=\lim_{y\to 0-}f'(y)/f(y)=\infty$. The first limit is (almost) obvious. The second limit exists because $(\ln f)'$ is increasing on I. In fact, for fixed $n>1$, we have that $f'(y)=\sum_{k\geq 1}k^2e^{k^2y}\geq n^2f(y)-n^2\sum_{k=1}^n k^2e^{k^2y}$, and so $f'(y)/f(y)\geq n^2-n^2\left(\sum_{k=1}^n k^2e^{k^2y}\right)/f(y)$ for $y<0$. Since $\lim_{y\to 0-}f(y)=\infty$, it follows that $\eta\geq n^2-n^2\left(\sum_{k=1}^n k^2\right)/\infty=n^2$. Therefore, $\eta=\infty$. It follows that $\varphi:=f'/f:\mathbb{R}_-^*\to(1,\infty)$ is a bijection.

Let us first determine the conjugate of $\ln f$ which will be needed to express the conjugate h^* of h. Since the equation $(\frac{d}{dy}[vy-\ln f(y)]=)\ v-\varphi(y)=0$ has the (unique) solution $y=\varphi^{-1}(v)\in I$ for $v>1$, we obtain that

$$(\ln f)^*(v)=\sup\{vy-\ln f(y)\mid y\in I\}=v\varphi^{-1}(v)-\ln\left[f(\varphi^{-1}(v))\right]\quad\forall v>1.$$

Because $(\ln f)^*$ is lsc, we have that

$$\begin{aligned}(\ln f)^*(1)&=\lim_{v\to 1+}(\ln f)^*(v)=\lim_{y\to-\infty}[y\varphi(y)-\ln f(y)]\\&=\lim_{y\to-\infty}\left[y\frac{1+\sum_{n\geq 2}n^2e^{(n^2-1)y}}{1+\sum_{n\geq 2}e^{(n^2-1)y}}-y-\ln\Bigg(1+\sum_{n\geq 2}e^{(n^2-1)y}\Bigg)\right]=0.\end{aligned}$$

Finally,

$$\begin{aligned}(\ln f)^*(v)&=\sup_{y\in I}[vy-\ln f(y)]=\lim_{y\to-\infty}[vy-\ln f(y)]=(\lim_{y\to-\infty}y)\left(v-\lim_{y\to-\infty}\frac{\ln f(y)}{y}\right)\\&=(-\infty)\left(v-\lim_{y\to-\infty}\frac{f'(y)}{f(y)}\right)=(-\infty)(v-1)=\infty\quad\forall v<1.\end{aligned}$$

Let us determine now the conjugate of h for $(u,v)\in\mathbb{R}^2$. Since $f(y)\in\mathbb{R}_+^*$ for $y\in\mathbb{R}_-^*$,

$$\begin{aligned}h^*(u,v) &= \sup_{(x,y)\in\operatorname{dom}h}[xu+yv-h(x,y)] = \sup_{y\in\mathbb{R}_-^*}\left(yv+\sup_{x\in\mathbb{R}}\left[xu-e^x[f(y)]^3\right]\right)\\ &= \sup_{y\in\mathbb{R}_-^*}\left(yv+[f(y)]^3\exp^*\left(\frac{u}{[f(y)]^3}\right)\right).\end{aligned}$$

Hence $h^*(u,v)=\infty$ for $u\in\mathbb{R}_-^*$ and $h^*(0,v)=\iota^*_{\mathbb{R}_-^*}(v)=\iota_{\mathbb{R}_+}(v)$. For $u\in\mathbb{R}_+^*$ we have that

$$\begin{aligned}h^*(u,v) &= \sup_{y\in\mathbb{R}_-^*}(yv+u(\ln u-3\ln[f(y)]-1)=u(\ln u-1)+\sup_{x\in\mathbb{R}}(yv-3u\ln[f(y)])\\ &= u(\ln u-1)+3u(\ln f)^*\left(\frac{v}{3u}\right).\end{aligned}$$

In conclusion,

$$h^*(u,v)=\begin{cases}u(\ln u-1)+3u(\ln f)^*\left(\frac{v}{3u}\right) & \text{if } v\geq 3u>0,\\ 0 & \text{if } u=0\leq v,\\ \infty & \text{if } u<0,\ \text{ or } v<0,\ \text{ or } 0\leq v<3u.\end{cases}$$

It follows that

$$\left\{(u,v)\in\mathbb{R}_+^*\times\mathbb{R}_+^*\mid v>3u\right\}=\operatorname{int}(\operatorname{dom}h^*)\subset\operatorname{dom}h^*=\{(u,v)\in\mathbb{R}_+\times\mathbb{R}_+\mid v\geq 3u\}.$$

Moreover, because $\nabla h(x,y)=e^x\,[f(y)]^3\cdot(1,3\varphi(y))$, we get

$$\partial h(\operatorname{int}(\operatorname{dom}h))=\nabla h(\mathbb{R}\times I)=\operatorname{int}(\operatorname{dom}h^*).$$

For $(u,v)\in\mathbb{R}^2$ consider the set

$$S(u,v):=\left\{(u_{k,l,m})_{k,l,m\geq 1}\subset\mathbb{R}_+\mid\sum_{k,l,m\geq 1}u_{k,l,m}=u,\quad\sum_{k,l,m\geq 1}(l^2+k^2+m^2)u_{k,l,m}=v\right\};$$

clearly, $S(u,v)=\emptyset$ for $(u,v)\notin\operatorname{dom}h^*$, $S(u,3u)=\{(u_{k,l,m})_{k,l,m\geq 1}\subset\mathbb{R}_+\mid u_{1,1,1}:=u,$ $u_{k,l,m}:=0$ otherwise$\}$ for $u\geq 0$ and $S(0,v)=\emptyset$ for $v>0$.

Applying Proposition 15 (v), for $(u,v)\in\operatorname{int}(\operatorname{dom}h^*)=\partial h(\operatorname{int}(\operatorname{dom}h))$ we have that

$$h^*(u,v)=\min\left\{\sum_{k,l,m\geq 1}u_{k,l,m}(\ln u_{k,l,m}-1)\mid(u_{k,l,m})_{k,l,m\geq 1}\in S(u,v)\right\},\tag{27}$$

the minimum being attained uniquely for $(\overline{u}_{k,l,m}):=\frac{u}{[f(y)]^3}e^{(k^2+l^2+m^2)y}$ $(k,l,m\geq 1)$, where $y<0$ is the solution of the equation $v/u=3f'(y)/f(y)$; hence $S(u,v)\neq\emptyset$ in this case. In the case $v=3u\geq 0$, as seen above, $S(u,3u)$ is a singleton and relation (27) holds, too; for $v>0$ $(=u)$ we have that $S(0,v)=\emptyset$ and $h^*(0,v)=0$.

Observe that the solution for the case $v=3u\geq 0$ is not obtained by using the (formal) method of Lagrange multipliers. Also note that even the solution for $(u,v)\in\partial h(\operatorname{int}(\operatorname{dom}h))$ can not be obtained from the results of J. M. Borwein and his collaborators because, even if ℓ^p-spaces can be regarded as $L^p(\Omega)$-spaces, the measure of Ω is not finite (and, even more, the corresponding linear operators are not continuous).

Acknowledgement. C.Z. wishes to thank his co-author for introducing him in entropy minimization problems related to Statistical Physics. C.Z. also benefited from discussions with C.V. on the use of Convex Analysis in Mechanics. Related to this, C.V. recalled the influence J. J. Moreau had on his vision on Mechanics. J. J. Moreau was a member of the jury of C. V.'s PhD thesis [9] defended at Université de Poitiers in 1973; they were colleagues for a while, and remained good friends. Unfortunately, C.V. died in November 2014.

Thanks go to Prof. M. Durea and Prof. D. Fortuné for their remarks on a previous version of the manuscript.

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